Applications of Walsh functions in communications

It is—with our advanced technology—no longer necessary to design communications equipment around a concept of trigonometric-based functions. A possible sophistication now is neatly netted with Walsh functions

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Communication theory was founded on the system of sine-cosine functions. A more general theory has become known more recently; it replaces the sine-cosine functions by other systems of orthogonal functions, and the concept of frequency by that of sequency. Of these systems, the Walsh functions are of great practical interest since they lead to equipment that is easily implemented by semiconductor technology. Filters, multiplexing equipment, and a voice analyzer/synthesizer have been built successfully for Walsh functions. Some interesting applications of electromagnetic Walsh waves have been found theoretically.

Traditionally, the theory of communication has been based on the complete, orthogonal* system of sine and cosine functions. The concept of frequency is a consequence of these functions, since frequency is defined as the parameter f in $\sin 2\pi ft$ and $\cos 2\pi ft$. The question arises whether there are other systems of functions on which theories of similar scope can be based, and that lead to equipment of practical interest. Since sine and cosine form a system of orthogonal functions, it is reasonable to investigate other systems of orthogonal functions.

Figure 1 shows three orthogonal systems: sine–cosine functions, Walsh functions, and block pulses, for which the normalized time $\theta = t/T$ is the variable. The block pulses are representative of several pulse shapes used for time multiplexing. The notations sal (i, θ) and cal (i, θ) are used here for the Walsh functions. (The letters *s* and *c* allude to the sine and cosine functions which are closely related to Walsh functions; the letters *al* are derived from the name Walsh.)[1-8]

Block pulses form an incomplete system; sine–cosine and Walsh functions form a complete system. Explicitly, the difference is that additional sine–cosine or Walsh functions may be drawn in Fig. 1 for $i = 5, 6, \ldots$ in the interval $-\frac{1}{2} \leq \theta < \frac{1}{2}$, but no other block pulses are orthogonal to the eight shown. Practically, the difference between complete and incomplete systems of functions is shown by the existence of elaborate theories based on sine–cosine functions for antennas, waveguides, and filters; no such theories exist for block pulses although used in communications much longer.

The Walsh functions in Fig. 1 assume the values +1 and −1 only—a useful feature if circuits are to be constructed with binary digital components. The functions cal (i, θ) and $\sqrt{2} \cos 2i\pi\theta$ are symmetric (even). The functions sal (i, θ) and $\sqrt{2} \sin 2i\pi\theta$ are asymmetric.

In Fig. 1, the parameter i in $\sqrt{2} \sin 2i\pi\theta$ and $\sqrt{2} \cos 2i\pi\theta$ gives the number of oscillations in the interval $-\frac{1}{2} \leq \theta < \frac{1}{2}$ (that is, the normalized frequency $i = fT$). One may interpret i as "one half the number of zero crossings per unit time" rather than as "oscillations per unit time." (The zero crossing at the left side, $\theta = -\frac{1}{2}$, but not the one at the right side, $\theta = +\frac{1}{2}$, of the time interval is counted for sine functions.)

The parameter i also equals one half the number of zero crossings in the interval $-\frac{1}{2} \leq \theta < \frac{1}{2}$ for Walsh functions. In contrast to sine–cosine functions the sign changes are not equidistant.† If, unlike Fig. 1, i is not an integer, then it equals "one half the average number of zero crossings per unit time." The term "normalized sequency" has been introduced for i, and $\varphi = i/T$ is called the nonnormalized sequency:

Sequency in zps = ½ (average number of zero crossings per second)

The general form of a sine function $V \sin (2\pi ft + \alpha)$ contains the parameters amplitude V, frequency f, and phase angle α. The general form of a Walsh function V sal $(\varphi T, t/T + t_0/T)$ contains the parameters amplitude V, sequency φ, the delay t_0, and time base T. The normalized delay, t_0/T, corresponds to the phase angle. The time base T is an additional parameter and it causes a major part of the differences in the applications of sine–cosine and Walsh functions.

* The two functions $f(j, x)$ and $f(k, x)$ in Fig. 1 are orthogonal in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ if the integral

$$\int_{-1/2}^{1/2} f(j, x)\, f(k, x)\, dx$$

is zero for $j \neq k$. They are orthogonal and normal or orthonormal if the integral equals 1 for $j = k$. A system of functions $\{f(j, x)\}$, orthogonal in a certain interval, is called complete if any function $F(x)$ quadratically integrable in that interval can be represented by a superposition of the functions $f(j, x)$ with a vanishing mean-square error.

† The first five Walsh functions look like heavily amplitude-clipped sine or cosine functions and have equidistant sign changes. This does not in general hold for functions with i larger than 2.

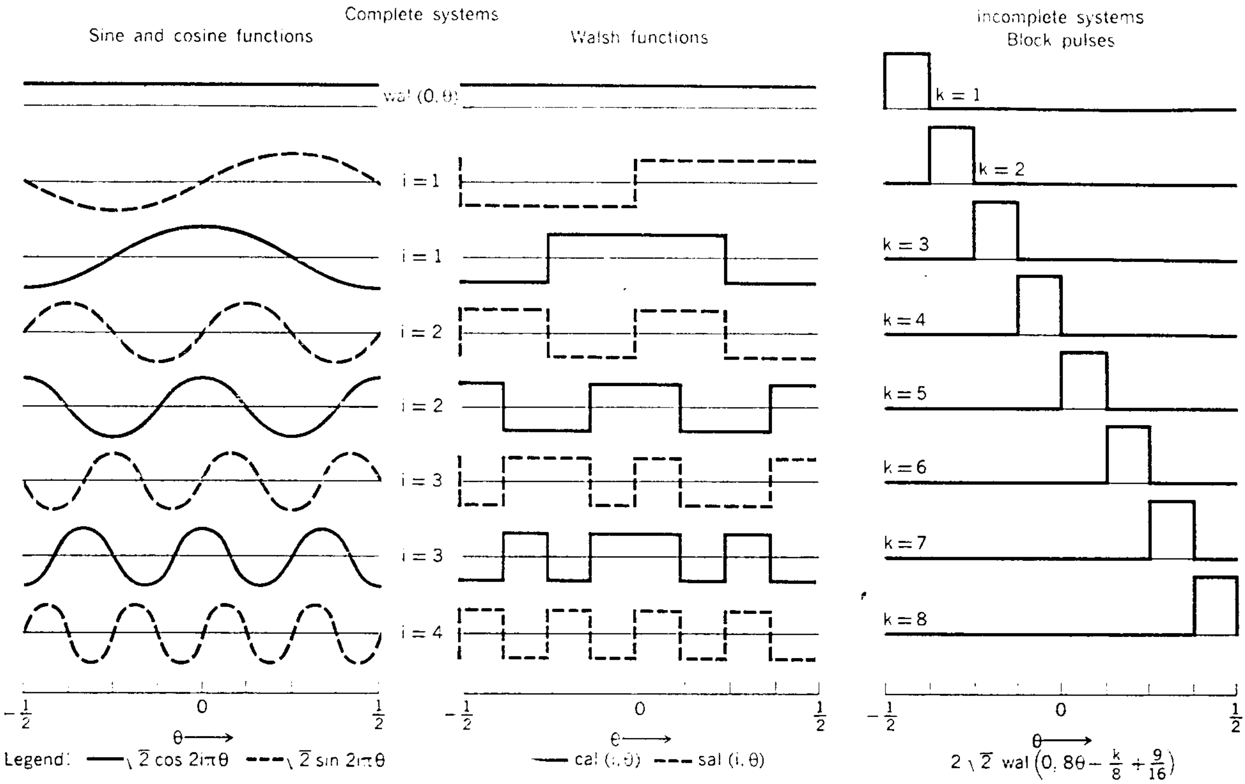


FIGURE 1. Orthonormal systems of functions.

Another alternative

So far, Walsh functions are the only known functions with desirable features comparable to sine–cosine functions for use in communications.* Development of semiconductor technology has imparted practical interest in them at this time. As an example of how this development has changed the approach to filter synthesis, consider the role of the capacitor and coil, which until recently were the most desirable components of filters. Such frequency-selective filters are linear, time-invariant, and thus a theory based on sine–cosine functions has indisputable advantages. But filters based on Walsh functions are linear and periodic time-variable. Generally speaking, the transition from sine–cosine functions to other complete systems of orthogonal functions means a transition from linear, time-invariant components and equipment to linear, time-variable components and equipment, which, of course, constitute a much larger class.

Figure 2 lists features of sine–cosine functions, Walsh functions, and block pulses. The mathematical theory of Walsh–Fourier analysis corresponds to the Fourier analysis used for sine–cosine functions. There is no theory of similar scope for block pulses, because they are incomplete.

* Walsh functions are closely related to Hadamard matrices. These matrices are orthogonal, consisting of square arrays of plus and minus ones, and are of the order 2^n. Other complete systems can be derived from Hadamard matrices of different rank.

Sine and cosine transforms of a function $F(\theta)$ are†

$$a_s'(\mu) = \int_{-\infty}^{\infty} F(\theta)\sqrt{2}\sin 2\pi\mu\theta\, d\theta$$
$$a_c'(\mu) = \int_{-\infty}^{\infty} F(\theta)\sqrt{2}\cos 2\pi\mu\theta\, d\theta \tag{1}$$

The corresponding sal and cal transforms of Walsh–Fourier analysis are defined by

$$a_s(\mu) = \int_{-\infty}^{\infty} F(\theta)\ \mathrm{sal}\,(\mu, \theta)\, d\theta$$
$$a_c(\mu) = \int_{-\infty}^{\infty} F(\theta)\ \mathrm{cal}\,(\mu, \theta)\, d\theta \tag{2}$$

$$F(\theta) = \int_{-\infty}^{\infty} [a_s(\mu)\ \mathrm{sal}\,(\mu, \theta) + a_c(\mu)\ \mathrm{cal}\,(\mu, \theta)]\, d\mu \tag{3}$$

where $\mu = \varphi T$ and $\theta = t/T$.

Walsh-function filters

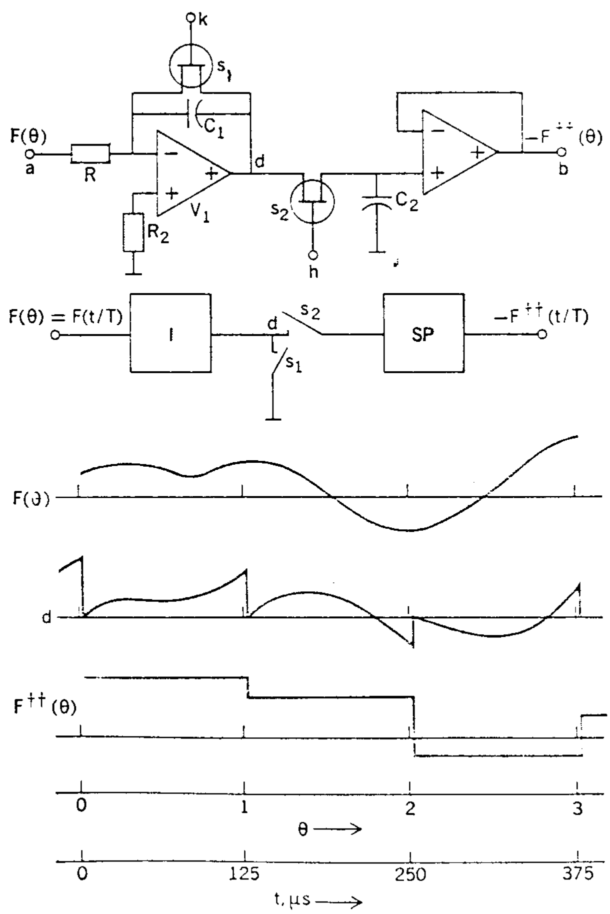
Figure 3 shows the block diagram, the time diagram, and a practical circuit of a sequency low-pass filter based

† The functions of Fig. 1 are defined in a finite interval but may be continued periodically to infinity. The parameter i is an integer and assumes denumerably many values. These functions are used for Fourier and Walsh–Fourier series expansions. The functions used for the Fourier and Walsh–Fourier transforms in Eqs. (1) and (2) are defined in the infinite interval. The parameter μ may be any real number and assumes nondenumerably many values.

	Sine Functions	Walsh Functions	Block Pulses
Parameters	Amplitude Frequency Phase angle . . .	Amplitude Sequency Delay Time base	Amplitude . . . Pulse position Pulse width
Mathematical theory	Fourier analysis	Walsh–Fourier analysis	. . .
Power spectrum	Frequency spectrum	Sequency spectrum	. . .
Filters	Time-invariable, linear	Periodically time variable, linear	Time variable, linear
Characterization	Frequency response of attenuation and phase shift	Sequency response of attenuation and delay	Attenuation as function of time
Multiplex	Frequency division	Sequency division	Time division
Modulation	Amplitude, phase, frequency modulation	Amplitude, time position, time base, code modulation	Amplitude, pulse position, pulse width modulation
Radiable	sin $2\pi ft$, cos $2\pi ft$	sal $\left(i, \frac{t}{T}\right)$, cal $\left(i, \frac{t}{T}\right)$	. . .

FIGURE 2. List of features and applications of sine–cosine functions, Walsh functions, and block pulses.

FIGURE 3. Sequency low-pass filter. Top to bottom: practical circuit, block circuit, time diagrams.



on Walsh functions. The input signal, $F(\theta)$, is transformed into a step function, $F\dagger\dagger(\theta)$, with steps of a certain width, by integrating $F(\theta)$ during an interval equal to the step width (see line *d*). The amplitudes of the steps are chosen so that $F\dagger\dagger(\theta)$ yields a least-mean-square approximation of $F(\theta)$. In addition, $F\dagger\dagger(\theta)$ is delayed with respect to $F(\theta)$ by one step width. The voltage obtained at the end of the interval is sampled by the switch s_2 and stored in the holding circuit *SP*. Immediately after sampling, the integrator is reset by s_1. If the width of the steps is 125 μs, $F\dagger\dagger(\theta)$ will have 8000 independent amplitudes per second. $F\dagger\dagger(\theta)$ may be considered to consist of a superposition of Walsh functions having 0 to 8000 zero crossings per second or a sequency between 0 and 4 kzps. The output signal of a frequency low-pass filter with 4-kHz cutoff frequency also has 8000 independent amplitudes per second. Hence, the sampling theorems of Fourier and Walsh–Fourier analysis permit the comparison of frequency and sequency filters.

Consider the multiplication theorems of sine and cosine shown in Fig. 4. The product of these two functions always yields a sum of two functions with argument $(k - i)\theta$ and $(k + i)\theta$. Let cos $i\theta$ and sin $i\theta$ represent carriers and let cos $k\theta$ and sin $k\theta$ represent Fourier components of a signal. The terms on the right sides of the multiplication theorems of Fig. 4 represent "lower" and "upper components" produced by amplitude modulation. Lower and upper sidebands are obtained if a carrier is amplitude-modulated by many rather than by one Fourier component. Hence, double-sideband modulation is a result of the multiplication theorems of sine–cosine functions.

Figure 4 also shows multiplication theorems for Walsh functions. The symbol $\oplus$ indicates an addition modulo 2: The numbers are written in binary form and added

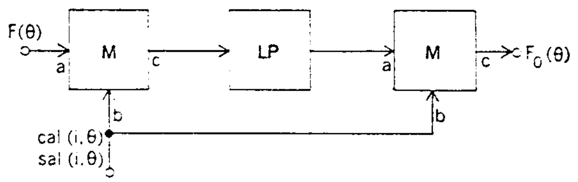
$$2\cos k\theta\ \cos i\theta = \cos(k-i)\theta + \cos(k+i)\theta$$
$$2\sin k\theta\ \cos i\theta = \sin(k-i)\theta + \cos(k+i)\theta$$
$$2\cos k\theta\ \sin i\theta = -\sin(k-i)\theta + \sin(k+i)\theta$$
$$2\sin k\theta\ \sin i\theta = \cos(k-i)\theta - \cos(k+i)\theta$$

$$\mathrm{cal}(k,\theta)\,\mathrm{cal}(i,\theta) = \mathrm{cal}\{k \oplus i,\theta\}$$
$$\mathrm{sal}(k,\theta)\,\mathrm{cal}(i,\theta) = \mathrm{sal}\{[i \oplus (k-1)]+1,\theta\}$$
$$\mathrm{cal}(k,\theta)\,\mathrm{sal}(i,\theta) = \mathrm{sal}\{[k \oplus (i-1)]+1,\theta\}$$
$$\mathrm{sal}(k,\theta)\,\mathrm{sal}(i,\theta) = \mathrm{cal}\{(k-1) \oplus (i-1),\theta\}$$
$$\mathrm{cal}(0,\theta) = \mathrm{wal}(0,\theta)$$

FIGURE 4. Multiplication theorems of sine–cosine and Walsh functions.

FIGURE 5. Sequency bandpass filter (A) and multiplier for Walsh functions (B).

A



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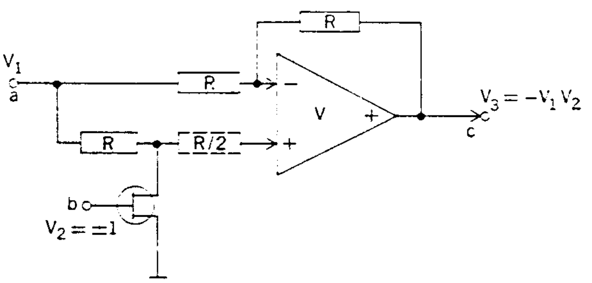
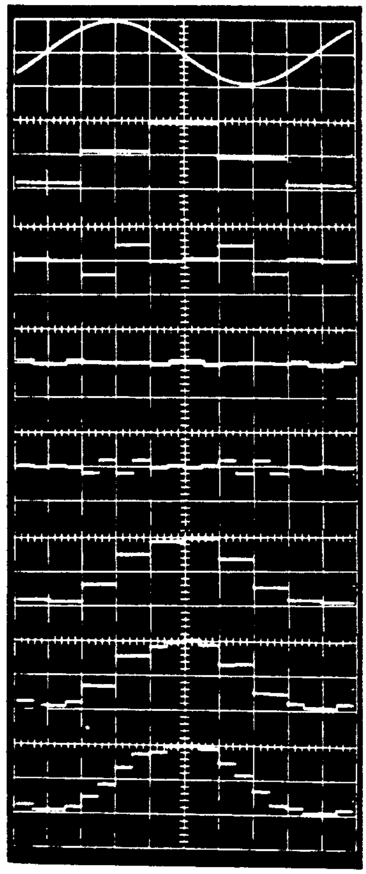




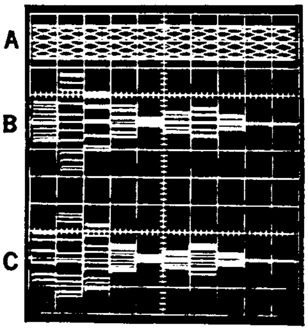


FIGURE 6 (left). Filtering of a sinusoidal voltage by various sequency filters; (A) is a sinusoidal function, frequency 250 Hz. Time base T = 1 ms; horizontal scale 0.5 ms/div. The following functions pass through the filters: (B) cal (φT, t/T), 0 ≤ φ < 1 kzps; (C) sal (φT, t/T), 0 < φ ≤ 1 kzps; (D) cal (φT, t/T), 1 kzps ≤ φ < 2 kzps; (E) sal (φT, t/T), 1 kzps < φ ≤ 2 kzps; (F) sum of B and C; (G) sum of B, C, and D; (H) sum of B, C, D, and F. (Courtesy Boeswetter and Klein)

FIGURE 7 (right). Amplitude spectra of sinsoidal voltages. Line (A) represents sinusoidal voltages, frequency 1 kHz, various phases; horizontal scale 0.1 ms/div. Lines (B) and (C) are amplitude spectra $a_c(\varphi T)$ and $a_s(\varphi T)$; time base T = 1.6 ms; horizontal scale 625 zps/div. (Courtesy Boeswetter and Klein)

according to the rules 1 ⊕ 0 = 0 ⊕ 1 = 1, 0 ⊕ 0 = 1 ⊕ 1 = 0 (no carry). The point is, the product of two Walsh functions yields only one Walsh function and not two. Let cal (*i*, *θ*) and sal (*i*, *θ*) represent carriers and let cal (*k*, *θ*) and sal (*k*, *θ*) represent Walsh–Fourier components of a signal. The amplitude modulation of a Walsh carrier yields only one component or only one (sequency) sideband.

A typical application of the multiplication theorems of Walsh functions is in the design of sequency-bandpass filters. Figure 5 shows the operating principle of such a bandpass. The input signal *F(θ)* is shifted in sequency by multiplication with a Walsh carrier, cal (*i*, *θ*) or sal (*i*, *θ*), and then passed through a sequency low-pass filter *LP* shown in Fig. 3. The filtered signal is subsequently shifted to its original position in the sequency domain by multiplication with the same Walsh carrier used to shift the input signal. Figure 5 also shows a typical multiplier for Walsh functions. Note that a signal *F(θ)* is multiplied by only +1 or −1. Multiplication by +1 leaves the signal unchanged; −1 reverses amplitude.

Sequency filters based on Walsh functions have been built by Boeswetter (Technische Hochschule, Darmstadt, W. Germany) for a voice analyzer and synthesizer, and also by Lueke and Maile (AEG–Telefunken, Research Department, Ulm, W. Germany) for a telephony multiplex system. In the latter application, a minimum crosstalk attenuation of about −60 dB was achieved in the stop bands. Such high-quality filters differ, of course, from the circuits shown in Figs. 3 and 5. Vandivere (Telcom Inc., McLean, Va.) has developed sequency filters for a signal analyzer.

Decomposition of voice signals by Walsh functions first seems to have been investigated theoretically by Sandy[9] in 1962. Another early investigator of sequency power spectra was Ohnsorg (Honeywell Inc., St. Paul, Minn.) whose work has not yet been published. Klein (Technische Hochschule, Darmstadt, W. Germany) has

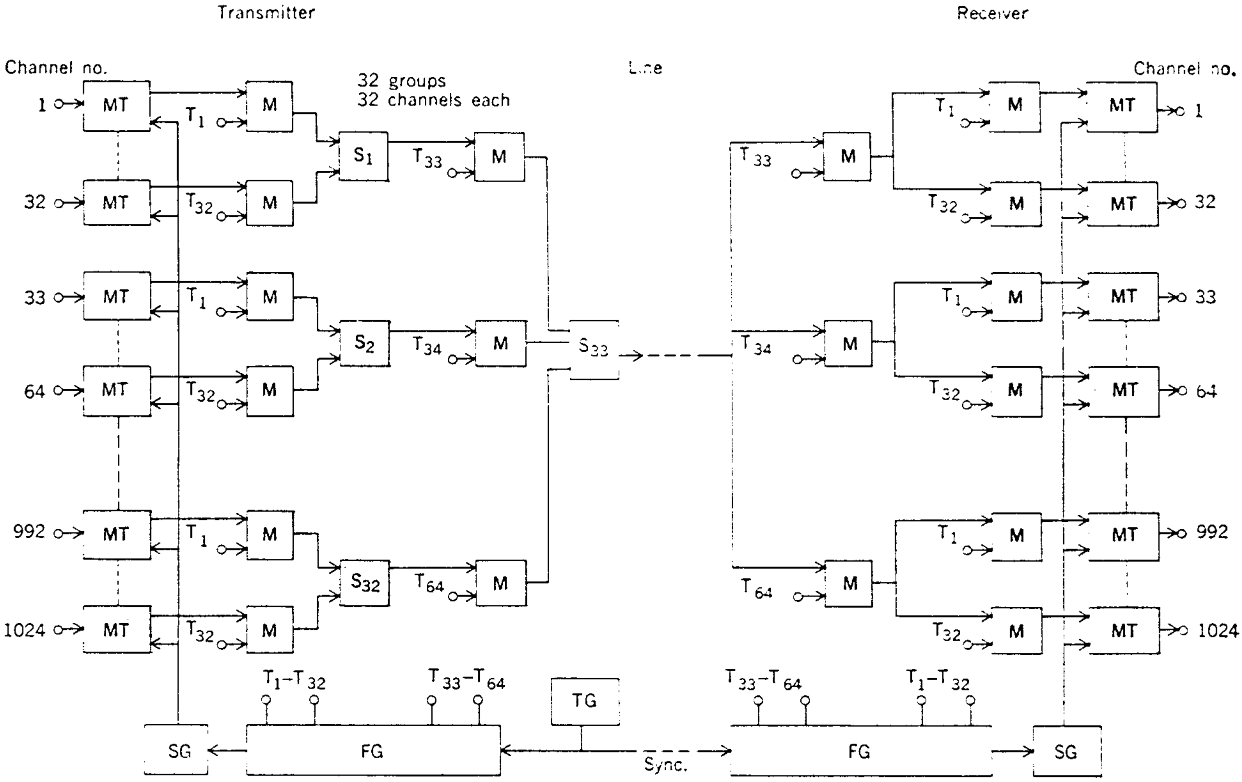


FIGURE 8. Block diagram of a sequency-multiplex system for 1024 telephony channels. Designations are: MT, sequency low-pass filter; M, multipliers; S, adder; FG, function generator; TG, clock pulse generator; SG, timing generator for the filter.

shown experimentally for some simple cases that voice signals have sequency formants just as they have frequency formants. Work on voice signals was also recently started by Elsner (Technische Hochschule, Braunschweig, W. Germany) and Strum (Mitre Corp., McLean, Va.). C. Brown (Systems Research Labs., Dayton, Ohio) is investigating signal processing techniques using sequency spectra for the purpose of detection and recognition of signals in noise, signal sorting, and signal parameter identification.

Figure 6 shows oscillograms of a sinusoidal voltage (A) and the voltages it produces at the output of various sequency low-pass filters (B, F, G, H) and sequency bandpass filters (C, D, E). Figure 7 shows sinusoidal voltages of fixed frequency and their sequency amplitude spectra $a_c(\mu) = a_c(\varphi T)$ and $a_s(\mu) = a_s(\varphi T)$.

Signal multiplexing

The multiplication theorems of the Walsh functions make signal multiplexing an attractive application. Figure 8 shows the principle of a sequency-multiplex system for 1028 telephony channels. Analog or digital signals are fed through sequency low-pass filters *MT* to multipliers *M*. For voice signals the bandwidth of these low-pass filters is $\Delta\varphi = 4$ kzps. Thirty-two carriers, T_1 to T_{32}, consisting of Walsh functions cal (i, θ) and sal (i, θ) are fed to the multipliers.* The time base of these functions equals $\Delta\varphi/2 = 125\ \mu$s. Each of the output voltages of the 32 multipliers is summed by the adders S_1 to S_{32}.

* Rules for the selection of carriers that avoid crosstalk and waste of sequency bandwidth can be derived from the multiplication theorems of the Walsh functions.[10]

These summed voltages may further be multiplied by another set of multipliers with other Walsh carriers, denoted T_{33} to T_{64} in Fig. 8. A sequency-multiplex system permits repeated sequency shifting just as a frequency multiplex system permits repeated frequency shifting.

The signals are separated at the receiver by multiplication with the same synchronized Walsh carriers used in the transmitter. The block diagram of Fig. 8 differs from that of a frequency-multiplex system only by the missing single-sideband filters. The circuitry inside the blocks is, of course, very different.

A sequency-multiplex system according to Fig. 8 has been developed by Lueke and Maile. The system is designed for 256 voice channels, of which three are fully completed.† More channels are presently being added by Huebner[12] of the West German Post Office Department in preparation for tests on post office lines. One of the tests will be the transmission of PCM voice signals via a scatter link, since, compared with time-division PCM, a gain of some 3 dB is predicted for this application. A Walsh-function tracking filter is used in the equipment of Lueke and Maile to establish synchronization between transmitter and receiver. The synchronization is good enough to yield a crosstalk attenuation of −57 dB, or better, in back-to-back operation. Crosstalk due to all

† Development of sequency-multiplex equipment was also started at ETH Zurich (Swiss Federal Institute of Technology, Institute for Advanced Electrical Engineering) and at the Research Institute of the West German Post Office Department in Darmstadt. Considerable theoretical work on PCM transmission by Walsh functions was done by Taki and Hatori[11] of Tokyo University, Japan. Experimental work has also been reported by Cox of the M.I.T. Instrument Laboratory, Cambridge, Mass.

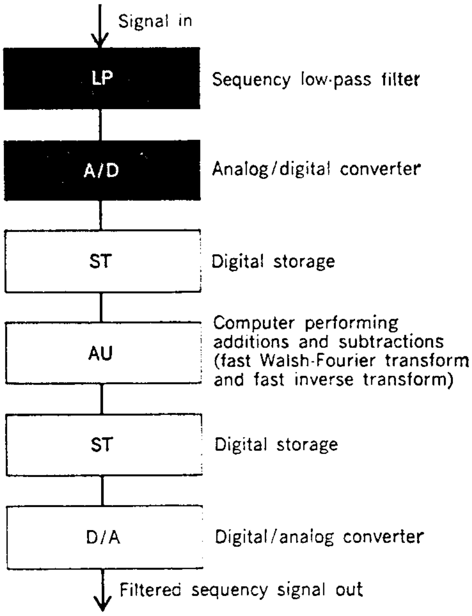


FIGURE 9. Digital sequency filter.

causes is −53 dB or better in back-to-back operation. Using a compandor (compressor plus expander), the apparent crosstalk attenuation could be made some −75 dB. These figures are, of course, in excess of that required for good PCM transmission.

Sequency-multiplex systems do not need single-sideband filters. The low-pass and bandpass filters required do not cause attenuation or delay distortions. These features are highly important for data transmission. All filters can be implemented by integrated circuit techniques. No individual tuning of the filters is required. The bandwidth of the filters is determined by the timing of the pulses that drive the switches s_1 and s_2 in Fig. 3; correct timing of the pulses replaces the tuning of filters; no temperature compensation is needed. The Walsh carriers can be produced by means of binary counters and gates. The only part in a sequency-multiplex system that requires tuning and temperature compensation is the clock pulse-generator.

Time-multiplex systems do not need single-sideband filters either, and they are also well suited for implementation by semiconductor technology. The advantage of a sequency-multiplex system here rests with reduced sensitivity to disturbances—brought about by two causes:

1. Only a fraction of all channels is active in a multiplex system at any one time. For instance, the activity factor of a telephone system does not exceed one quarter even during peak traffic.* Hence, the power amplifiers are not used three quarters of the time and the average useful signal power is reduced correspondingly in a time-multiplex system. Sequency-multiplex systems yield a higher average signal power for equal peak power. Particularly advantageous is that the average power can be maintained approximately constant by means of automatic gain-control amplifiers when the activity factor drops during low-traffic hours. It is also noteworthy that the sequency bandwidth of a sequency-multiplex signal is not changed by a nonlinear compressor or expander characteristic; hence, sequency-multiplex signals can readily be passed through instantaneous compandors.

* The activity factor gives the number of channels actually carrying signals. It is smaller than one during peak traffic because one party must listen while the other one talks, because there are idle periods in conversations, because a channel becoming available is not immediately used, etc.

2. Digital signal errors during transmission through telephone lines are mainly caused by pulse-type disturbances. Time division is more susceptible to these disturbances than frequency or sequency division since a particular block pulse may be changed appreciably by a disturbing pulse, although the preceding and the following pulses are not changed at all. In the case of frequency or sequency multiplexing, many sine–cosine or Walsh pulses are transmitted simultaneously. The energy of a disturbing pulse is thus spread over many signal pulses. Only considerable energy can disturb signal pulses that are quantized. Measurements with binary sine–cosine pulses have yielded—and theoretically Walsh pulses should yield—error rates 100 times smaller than for block pulses having the same average power.

There are several additional—although not so important—differences between time- and sequency-multiplex systems. For example, equipment for time division is often less expensive; sequency multiplexing is more adaptable to the problems of communication networks and makes it easier to mix voice and data signals with different average power.

Digital filtering and multiplexing

One of the most promising aspects of Walsh functions is the ease with which filters and multiplex equipment can be implemented as digital circuits. The reason is that numerical Walsh–Fourier transformation and numerical sequency shifting of signals require summations and subtractions only. In the case of sine–cosine functions, the corresponding operations require multiplications with irrational numbers. The simplification of numerical computations by the use of Walsh functions has been recognized by many scientists.[12-17]

Figure 9 shows the block diagram of a digital filter based on Walsh functions. The input signal passes first through a sequency low-pass filter *LP* that transforms it into a step function. This step function is sampled and the samples are transformed into numbers by an analog/digital converter. A series of these numbers is stored in a digital storage *ST*. A Walsh–Fourier transform of this series is obtained by performing certain additions and subtractions in the arithmetic unit *AU*. Some or all of the obtained coefficients, that represent sequency components, may be suppressed or altered—in effect, a filtering process. An inverse Walsh–Fourier transform yields the filtered signal as a series of numbers. These numbers are stored in a second digital storage *ST* and transformed into an analog signal by digital/analog converter *D/A*. Since there is a fast Walsh–Fourier transform just as there is a fast Fourier transform, the arithmetic operations in a digital sequency filter are not only simpler than in a digital frequency filter but can be performed faster.[10, 18-21]

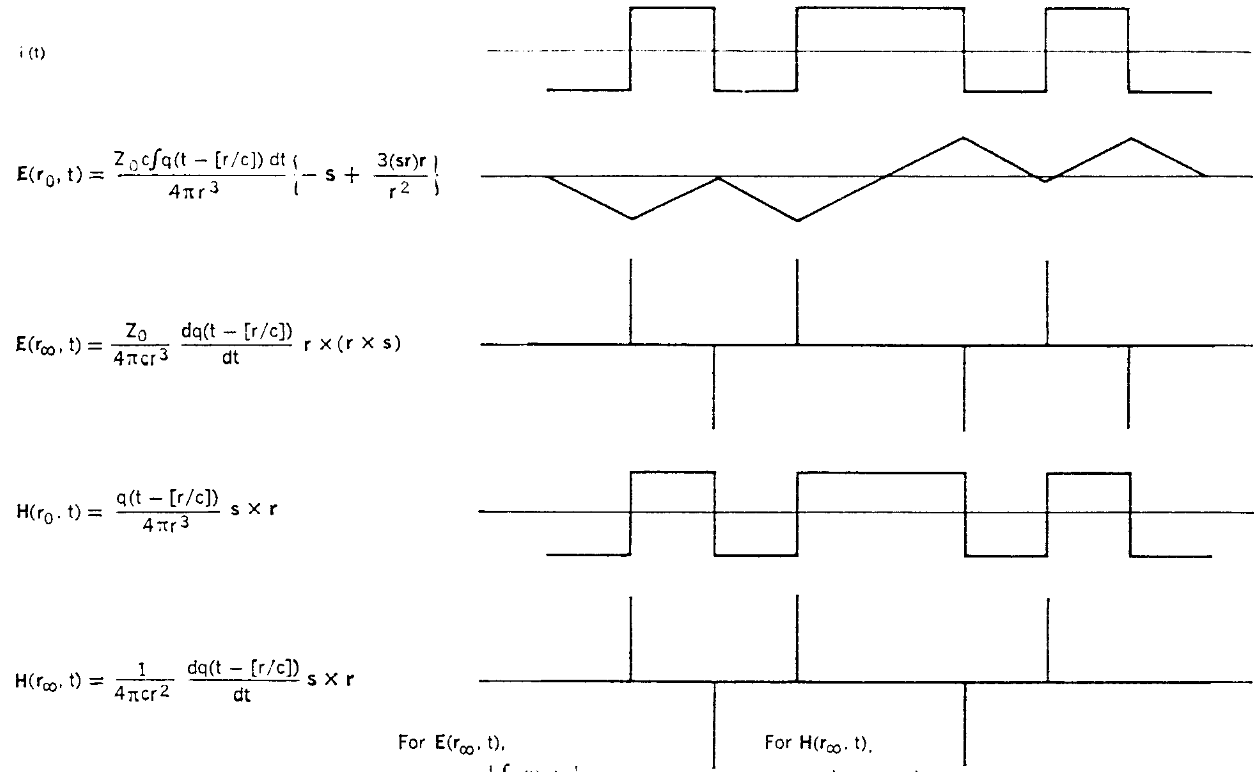
Voice signals are functions of the one variable, time. A black-and-white photograph is a function of two space variables, and a black-and-white television picture is a function of two space variables and time. Digital filters may be applied to signals that are functions of two or three variables by using a two- or three-dimensional

$$\mathbf{E}(\mathbf{r}_0, t) = \frac{Z_0 c \int q(t - [r/c])\, dt}{4\pi r^3} \left\{ -\mathbf{s} + \frac{3(\mathbf{s}\mathbf{r})\mathbf{r}}{r^2} \right\}$$

$$\mathbf{E}(\mathbf{r}_\infty, t) = \frac{Z_0}{4\pi c r^3} \frac{dq(t - [r/c])}{dt} \mathbf{r} \times (\mathbf{r} \times \mathbf{s})$$

$$\mathbf{H}(\mathbf{r}_0, t) = \frac{q(t - [r/c])}{4\pi r^3} \mathbf{s} \times \mathbf{r}$$

$$\mathbf{H}(\mathbf{r}_\infty, t) = \frac{1}{4\pi c r^2} \frac{dq(t - [r/c])}{dt} \mathbf{s} \times \mathbf{r}$$



For $\mathbf{E}(\mathbf{r}_\infty, t)$, $r^2 >> c^2 \left| \frac{\int q(t)\, dt}{dq(t)/dt} \right|$

For $\mathbf{H}(\mathbf{r}_\infty, t)$, $r >> c \left| \frac{q(t)}{dq(t)/dt} \right|$

FIGURE 10. Electric and magnetic field strengths in the near and wave zones due to a current q(t) fed into a Hertzian dipole. The functions on the right show the time variations caused by a Walsh-shaped current q(t).

Walsh–Fourier transform.* This has been done by Pratt, Kane, and Andrews[18] for functions of two variables. Roth and Lueg (Technische Hochschule, Aachen, W. Germany) as well as Held (Technische Hochschule, Darmstadt, W. Germany) and Klein have also recently started to develop digital sequency filters. A very general method for digital signal filtering—not restricted to Walsh functions—was devised by Andrews[22] and Caspari (International Telephone and Telegraph Corporation, Electro-Physics Laboratories, Hyattsville, Md.).

Digital multiplexing is a straightforward extension of digital filtering. Rather than multiplying signals and Walsh carriers represented by voltages, one multiplies signals and Walsh carriers represented by series of numbers. Multiplexed signals are again represented by a series of numbers that may be transmitted by any digital communication equipment. The promising feature of such a digital-sequency-multiplex system is that it has essentially the same immunity to disturbances as the previously discussed analog-sequency-multiplex equipment, but is compatible with existing transmission equipment.

Electromagnetic Walsh waves

At the present time, sinusoidal electromagnetic waves exclusively are used for radio communication. Such waves are characterized by a sinusoidal variation with time of the electric and magnetic field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$. These field strengths are produced by feeding a sinusoidal current into the antenna.

Figure 10 shows the time variation of electric and magnetic field strengths for a Walsh-shaped current $q(t)$ fed into a Hertzian dipole. A typical current is shown in the first line. $\mathbf{E}(\mathbf{r}_0, t)$ and $\mathbf{H}(\mathbf{r}_0, t)$ are the field strengths in the near zone, $\mathbf{E}(\mathbf{r}_\infty, t)$ and $\mathbf{H}(\mathbf{r}_\infty, t)$ the field strengths in the wave zone. Z_0 (377 ohms) is the wave impedance of free space, c the velocity of light, r the distance between dipole and observation point, **r** the vector from dipole to the observation point, and **s** the dipole vector. The spatial variation of **E** and **H** depends on the vectors **s** and **r**; it is the same for sine, Walsh, and other waves. The time variation depends solely on the current $q(t)$ fed into the Hertzian dipole. This time variation is plotted in Fig. 10 on the left for the near zone and the wave zone of **E** and **H**. In the wave zone, one obtains Dirac delta functions for **E** and **H** that are located at the jumps of $q(t - r/c)$. The time variation of **H** in the near zone equals $q(t - r/c)$; the time variation of **E** equals $\int q(t - r/c)\, dt$.

The wave zone of **E** and **H** is the region in which the distance r between dipole and observation point satisfies the conditions shown in the bottom line of Fig. 10. In the near zone, the "much larger" signs in the inequalities are replaced by "much smaller" signs. These conditions have the more familiar form $r >> \lambda$ or $r << \lambda$ for sinusoidal currents $q(t) = I \sin 2\pi f t = I \sin (2\pi c t / \lambda)$.

A receiver always receives the sum of the field strengths

* Analog filters based on sine–cosine functions can be implemented for functions of two space variables by optical means. Analog filters based on Walsh functions can be implemented with relative ease for functions of two space variables by resistors and operational amplifiers; their implementation for functions of two space variables and time, or three space variables with and without the time variable, is possible but expensive.

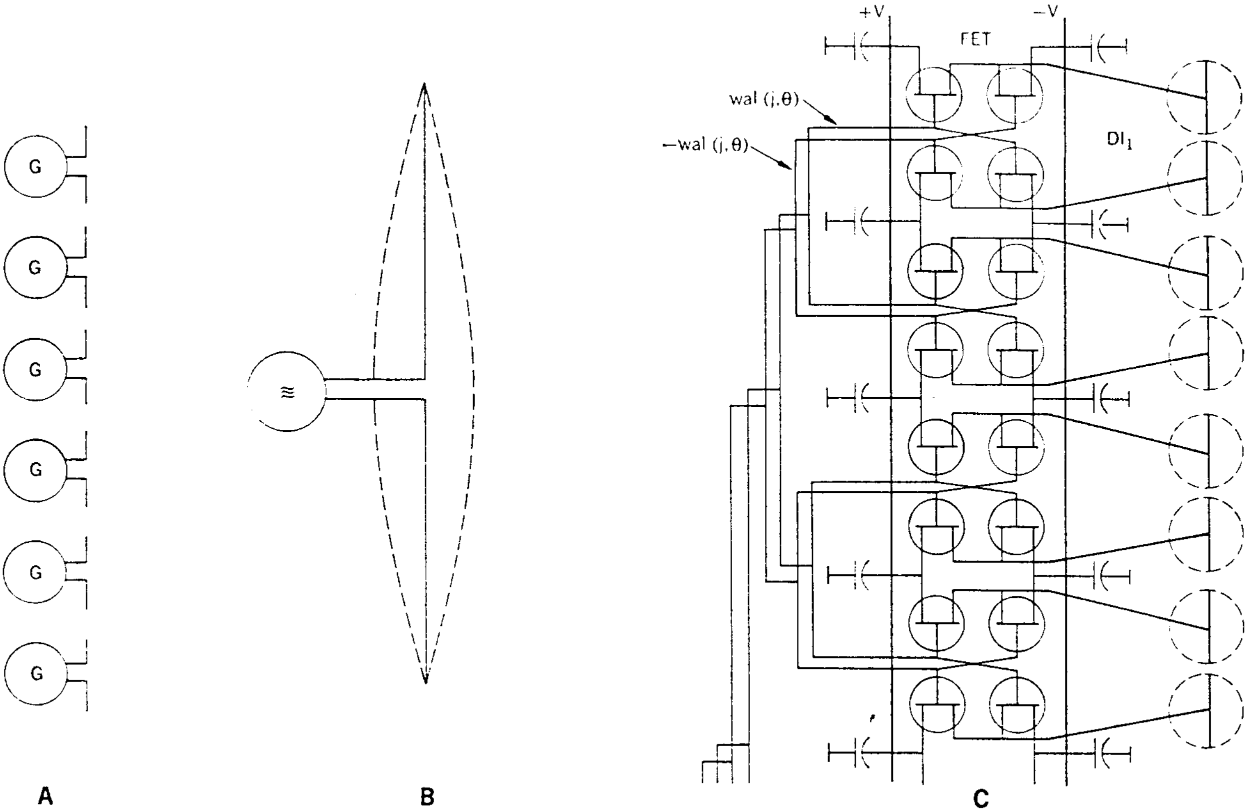
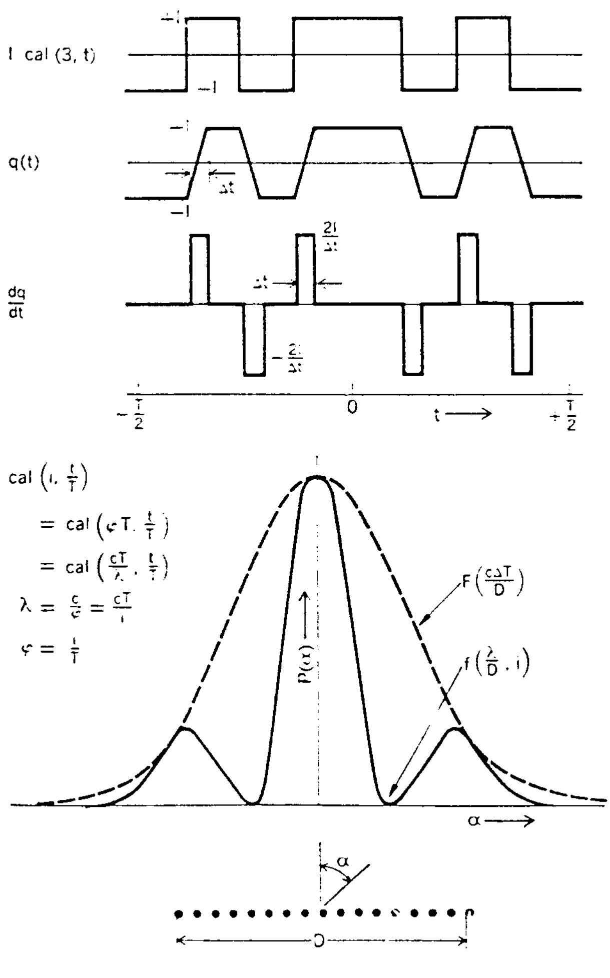


FIGURE 11. A—General array of Hertzian dipoles. B—λ/4 dipole for sine waves. C—Array of Hertzian dipoles and power amplifiers for Walsh waves [wal (2j, θ) = cal (j, θ)].

FIGURE 12. Beam width of an antenna for Walsh waves.



from the near and wave zones. $\mathbf{E}(\mathbf{r}_\infty, t)$ and $\mathbf{H}(\mathbf{r}_\infty, t)$ decrease proportionally with $1/r$, whereas $\mathbf{E}(\mathbf{r}_0, t)$ and $\mathbf{H}(\mathbf{r}_0, t)$ decrease proportionally with $1/r^3$ and $1/r^2$. Hence, a comparison of the field strengths of the near and the wave zones allows the distance of the transmitter to be determined. This determination of course, requires the time variation of the near- and the wave-zone components to be different in order to distinguish between them. A sinusoidal current $q(t)$ produces sinusoidal near-zone and wave-zone components since the differential as well as the integral of a sine function is simply a time-shifted sine function.

A Hertzian dipole radiates vanishingly little power. Many such dipoles may be used simultaneously to radiate more power. Figure 11(A) shows several Hertzian dipoles, each fed by one generator. In the case of sinusoidal waves, one may feed many Hertzian dipoles by a single high-power generator using standing waves in a half-wave or similar dipole [Fig. 11(B)]. This is also possible but not economical for Walsh waves. Since a generator for Walsh-shaped currents consists of switches that feed positive or negative currents into the dipole, it is better to feed many Hertzian dipoles through many switches as shown in Fig. 11(C). These switches must feed constant currents and not constant voltages to the dipoles. Current feed is automatic if the switches are semiconductors.

The Hertzian dipoles *DI* in Fig. 11(C) are circular areas of conducting material. They do not have to be arranged along a line as in Fig. 11(C), but may be arranged in two dimensions. Such a two-dimensional arrangement, however, cannot be used for antennas that use the standing-wave principle such as the one in Fig. 11(B).

The deviations of the antenna current $q(t)$ from the ideal shape shown in Fig. 10 must be taken into account

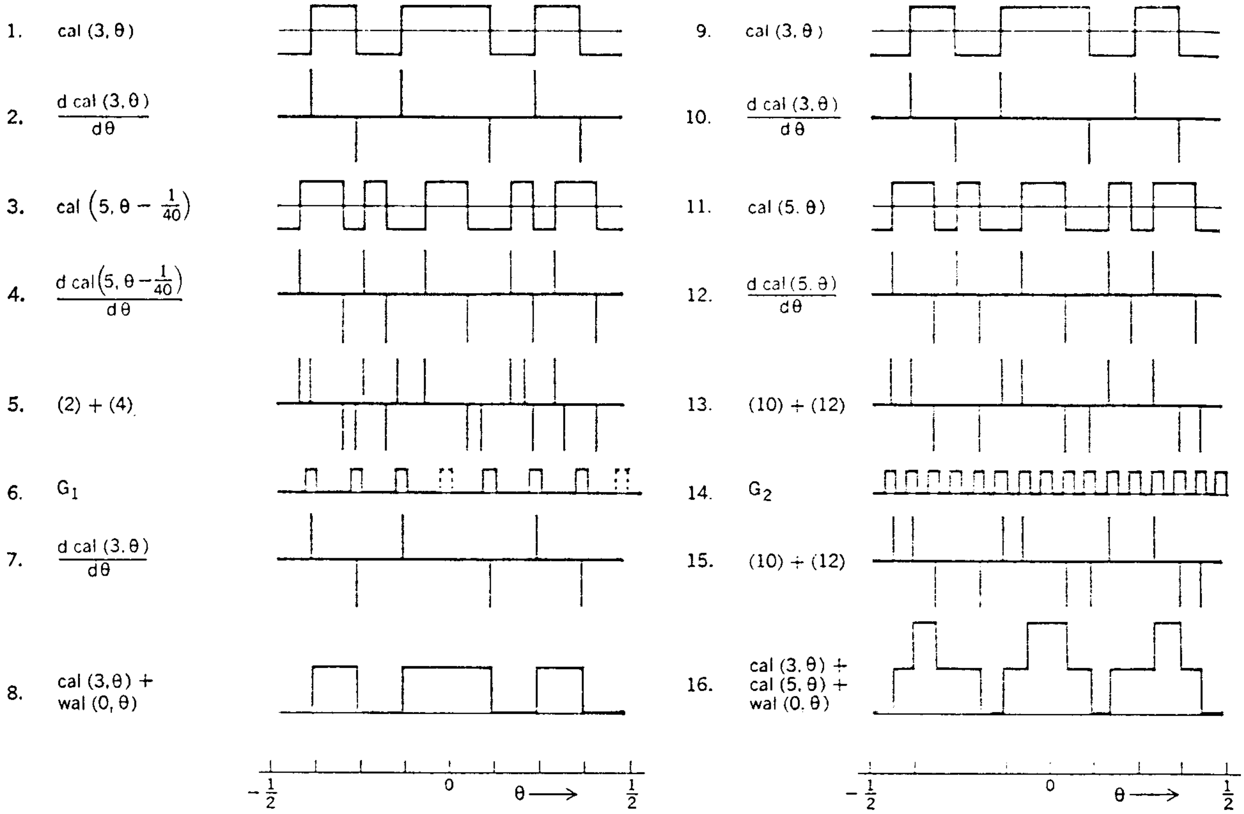


FIGURE 13. Principle of the separation of two Walsh waves in mobile radio communication.

if the directional characteristic of an antenna, consisting of many Hertzian dipoles, is to be determined. For example the idealized current, I cal (3, t) shown on the first line of Fig. 12 is not achievable since it is not possible to switch back and forth between $+I$ and $-I$ in zero time. The current $q(t)$ with finite switching times Δt is more realistic. The time differential dq/dt of the current $q(t)$ is shown in the third line. Narrow, rectangular pulses with finite amplitude—not delta pulses—result.

A row of Hertzian dipoles is shown at the bottom of Fig. 12. The length of this row is D; each dipole is short compared to $c\Delta t$. Let a current $q(t)$, as shown in the second line of Fig. 12, be fed into each dipole. A radiation diagram $P(\alpha)$ gives the average radiated power in the wave zone as function of the angle α. A typical radiation diagram is shown in Fig. 12 by the solid line above the row of dipoles. The dashed line is the envelope of the main lobe and the sidelobes. This dashed line is a function of $c\Delta t/D$, where c denotes the velocity of light and Δt the switching time of the current $q(t)$. A reduction of Δt makes the dashed line approach zero for angles $\alpha \neq 0$, and the antenna radiates into a very narrow angle for small Δt.

The location of the zeros of the solid curve in Fig. 12 depends on the ratio λ/D and the normalized sequency i of the Walsh waves. (λ is the average wavelength.) The following definitions apply to cal (i, t/T):

$$\text{cal}\,(i,\ t/T) = \text{cal}\,(\varphi T,\ t/T) = \text{cal}\,(cT^2/\lambda,\ t/T)$$

where $\lambda = c/\varphi = cT/i$ = average wavelength, $\varphi = i/T$ = sequency, and $i = 1, 2, \cdots$ = normalized sequency.

The directivity of an antenna is determined by the ratio λ/D for sine waves. In the case of Walsh waves it is determined by the ratios λ/D and $c\Delta t/D$, and by the normalized sequency i. The important point is that a reduction of Δt yields a significantly better directivity without a need to decrease λ or increase D. These results not only hold true for a row of dipoles in Fig. 12, but also for other antenna shapes, particularly parabolic reflectors. (D represents the reflector diameter.)

Maintaining orthogonality after time shifting

It is known that orthogonal functions—for example, voltages or field strengths varying with time—can be separated. It is not necessary that this separation be by frequency or time division; the more general orthogonal division is sufficient.[10] Hence, a point-to-point transmission is perfectly possible with Walsh waves. Mobile communication is more difficult, since waves radiated from various transmitters show various time shifts at the receiver due to the propagation times of the waves. These unknown time shifts generally destroy the orthogonality of the received functions. Up to now, the only known exception has been sine waves. A time shift destroys only the orthogonality between sine and cosine functions of the same—not different frequency.*

Walsh waves in the wave zone are a second exception† and may also be separated regardless of any time shift.

* The relation $\sin(\omega t + \gamma) = \sin\gamma \cos\omega t + \cos\gamma \sin\omega t$ shows that a time-shifted sine function consists of not-shifted sine and cosine functions with the same frequency. Any function orthogonal to $\sin\omega t$ and $\cos\omega t$ is thus orthogonal to $\sin(\omega t + \gamma)$.

† These waves have the shape of differentiated Walsh functions according to Fig. 10 rather than the shape of the Walsh functions.

Figure 13, line one, shows the function cal $(3, \theta)$ = cal $(3, t/T)$; line two shows its differential, which represents the time variation of the electric and magnetic field strength in the wave zone—Dirac pulses at the jumps of cal $(3, \theta)$. (The deviation from this theoretical shape is neglected here, just as it is usual to neglect other than idealized, infinitely long sine functions.)

Line three shows another Walsh function, cal $(5, \theta - 1/40)$, time-shifted relative to cal $(3, \theta)$; line four shows its differential; line five shows the sum of lines two and four. This sum is received if one transmitter radiates the wave cal $(3, \theta)$ and another the wave cal $(5, \theta)$. The voltage produced in the receiving antenna also varies with time according to line five. As shown by line six, a gate permits those pulses to pass that arrive at the proper time; the pulses of line seven are thus derived from those of line five. These are the same pulses as those of line two. Hence, the desired signal is separated from the non-desired one.* Integration of the pulses of line seven yields the function cal $(3, \theta)$ and a superimposed dc component in line eight. The dc component is of no consequence.

Lines nine to 16 in Fig. 13 show what happens if cal $(3, \theta)$ and cal $(5, \theta)$ are not time-shifted relative to each other. This case cannot occur if the Dirac pulses in lines two and four as well as the gating intervals in line six are infinitely short, but it is important for the practical pulses of finite duration. (More precisely, the probability of this case to occur is zero.) Lines nine to 13 correspond to one to five except for the time shift. The gate, however, opens 16 times according to line 14. Integration of the pulses passed (line 15) yields the sum, cal $(3, \theta)$ + cal $(5, \theta)$ + dc component wal $(0, \theta)$ [equals cal $(0, \theta)$] in line 16. Correlation of this sum with a sample function cal $(3, \theta)$ suppresses the components cal $(5, \theta)$ and wal $(0, \theta)$. The general rule for the opening times of the gate is as follows:

Let an arbitrary number of transmitters radiate Walsh waves cal $(i_1, t/T)$, cal $(i_2, t/T)$, $\cdots$, and let the normalized sequencies $i_1, i_2, \cdots$ assume values from 1 to 2^k only (k = integer). The gate must open periodically 2×2^k times during the time T and allow pulses to pass. Example: i_1 equals 3 and i_2 equals 5 in Fig. 13, line nine. These numbers are between 1 and $8 = 2^3 = 2^k$. Hence, the gate of line 14 must open periodically $2 \times 8 = 16$ times during the time T (the interval $-T/2 \leq t < +T/2$ corresponds to the interval $-1/2 \leq \theta = t/T < +1/2$). The time interval during which the gate stays open should be about as wide as the received pulses.

* The problem of separation is essentially the same as for synchronous reception of sinusoidal waves. Walsh waves of different sequency can be distinguished and thus separated like sine waves of different frequency. On the other hand, two waves caused by antenna currents I sal (i, θ) and I cal (i, θ) cannot be distinguished without a synchronization signal just as two waves caused by antenna currents I sin ωt and I cos ωt cannot be distinguished without a synchronization signal.

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