Multiple-Access Performance Limits with Time Hopping and Pulse Position Modulation

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ABSTRACT—The use of spread spectrum time hopping in combination with pulse position modulation for multiple access communications is studied. The multiple access performance is described in terms of the number of users supported by the system for a given bit error rate and bit transmission rate. Expressions for the maximum number of users and maximum transmission capacity in bits per second are found. Asymptotic values for these quantities are derived.

I. INTRODUCTION

Spread Spectrum multiple-access (MA) communication using time hopping (TH) modulation and impulse signal technology was proposed in [1]. This communication technique is called impulse radio (IR) [2]. Impulse radio modulation uses impulse signal technology to generate ultra-wideband (UWB) communication signals that consist of trains of time-shifted subnanosecond impulses. Data is transmitted using pulse-position-modulation (PPM) at a rate of many pulses per symbol, and MA capability is achieved using spread spectrum time hopping (TH). Impulse radio promises to be a viable technique to build relatively simple and low-cost, low-power transceivers that can be used for short range, high speed MA communications over the multipath indoor wireless channel [3].

In [1] the MA performance of IR assuming free space propagation conditions and additive white Gaussian noise (AWGN) was studied. The analysis assumed that binary PPM signals based on binary pulse-position-modulation are shift-coherent detected using a single-channel correlation receiver. In [5] the use of block-waveform encoding PPM signals to increase the number of users supported by the system for a given MA performance and bit transmission rate was investigated. A ten-fold increase in the number of users was shown to be achievable using a receiver of moderate complexity. In this paper we remove the receiver complexity constraint, and proceed to derive expressions for the maximum number of users and maximum transmission capacity in bits per second, calculating their respective asymptotic values.

II. Channel, signals and multiple-access interference models

A. Channel and impulse signal models

The model assumed is a free-space propagation channel impaired with AWGN. The transmitted impulse is $w_n(t) \triangleq \int_{-\infty}^{\infty} w(\xi) d\xi$ and the received signal is $w(t) + n(t)$. The noise $n(t)$ is AWGN with two-sided power density $\frac{N_0}{2}$. The signal $w(t)$ is the basic subnanosecond impulse used to convey information. It has duration $T_w$ and energy $E_w = \int_{-\infty}^{\infty} |w(t)|^2 dt$. The normalized signal correlation function of $w(t)$ is $\gamma_w(\tau) \triangleq (1/E_w) \int_{-\infty}^{\infty} w(t)w(t-\tau)dt > -1 \forall \tau$. We define $\gamma_{min} = \gamma_w(\tau_{min})$ as the minimum value of $\gamma_w(\tau)$, $\tau \in [0, T_w]$.

B. PPM signals

The PPM signals consist of $N_s$ time-shifted impulses

$$S_j(t) = \sum_{k=0}^{N_s-1} w(t-kT_T-\delta_k^j), \quad j = 1, 2, \ldots, M.$$

Notice that information is conveyed exclusively in the sequence of time shifts $\{\delta_k^j; k = 0, 1, 2, \ldots, N_s - 1\}$, with $\delta_k^j \in \{0 = \tau_1 < \tau_2 < \ldots < \tau_N\}$ for some $N \geq 2$. In IR, the impulse duration satisfies $T_T \ll T_J$, where $T_J$ is the time shift value corresponding to the frame period. A single symbol waveform has duration $T_T \approx N_T J$, with $N_T \gg 1$. The M-ary symbol rate is $R_s = T_T^{-1}$. The signals $\{S_j(t)\}$ have correlation values

$$R_{ij} = \int_{-\infty}^{\infty} S_i(\xi) S_j(\xi) d\xi = E_w \sum_{k=0}^{N_s-1} \gamma_w(\delta_k^i - \delta_k^j),$$

since for $k \neq l$ the pulses are non-overlapping. The energy in the $j^{th}$ signal is $E_j = R_{ii} = N_s E_w$, and the normalized correlation value is

$$\alpha_{ij} \triangleq \frac{R_{ij}}{E_S} = \frac{1}{N_s} \sum_{k=0}^{N_s-1} \gamma_w(\delta_k^i - \delta_k^j) \geq \gamma_{min} \forall \tau.$$

This approach is analogous to the one used in [4].

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3The range of frequencies occupied by the UWB impulse goes from a few hundreds of Kilohertz up to a few Gigahertz.
As in [5], we will work with equally correlated (EC) signals with

\[ a_{ij} = \lambda = \frac{1 + \gamma_w(\tau_2)}{2} > 0 \text{ for } N_s, >> 1 \]

with \( 0 < \tau_2 < T_w \).

C. TH PPM signals

Each user's signal is composed of a sequence of fast-hopped frame-shifted versions of one of the \( M \) possible PPM symbol waveforms \( \{ S_{\nu}(t) \} \)

\[ x^{(\nu)}(t) = \sum_{m=0}^{\infty} S_{m}^{(\nu)}(t - mN_sT_f - C^{(\nu)}_m(t)), \]

where the superscript \( \nu \), \( 1 \leq \nu \leq N_u \) indicates user-dependent quantities, \( N_u \) is the number of users, \( m \) indexes the transmitted symbols, \( S_{m}^{(\nu)} \in \{1, 2, \ldots, M\} \) is the \( m \)-th transmitted symbol, and

\[ C^{(\nu)}_m(t) = \left( \begin{array}{c} (m+1)N_s-1 \\ k=mN_s \end{array} \right) T_c c^{(\nu)}_k(t - kT_f), \]

\[ \phi(t) = \left\{ \begin{array}{ll} 1, & \text{if } 0 \leq t \leq T_f \\ 0, & \text{otherwise} \end{array} \right. \]

Here \( T_c \) is a time-shift value and \( \{c^{(\nu)}_k\} \) is the pseudo-random time-hopping sequence assigned to user \( \nu \), with \( 0 \leq c^{(\nu)}_k \leq N_h \) for some integer \( N_h \). The \( \{c^{(\nu)}_k\} \) has periodicity \( N_p \), i.e., \( c^{(\nu)}_{k+N_p} = c^{(\nu)}_k \) for \( k, l, N_p \) integers.

D. Multiple-access interference model

In the present analysis, performance computation is based on signal-to-noise ratios (SNR) averaged over random TH sequences \( \{c^{(\nu)}_k\} \) and random asynchronous transmission times \( \{\tau^{(\nu)}_m\} \). To this end the following assumptions were made:

(a) The elements \( \{c^{(\nu)}_k\} \) for \( \nu = 1, 2, \ldots, N_u \) and for all \( k \), are independent, identically and uniformly distributed on the interval \([0, N_h]\). Furthermore, we assume that \( N_s \leq N_p \).

(b) The transmission time differences \( \tau^{(\nu)}_m \text{ (mod } T_f) \), \( \nu = 2, \ldots, N_u \), are independent, identically and uniformly distributed on \([0, T_f]\).

(c) We assume that the received monocycle waveform satisfies the relation \( \int_{-\infty}^{\infty} w(t)dt = 0 \).

III. System performance

A. Multiple-access performance

When \( N_u \) transmitters are active and the receiver wishes to determine the data modulating transmitter \( \nu = 1 \), the received signal \( r(t) \) can be viewed as

\[ r(t) = A^{(1)} S_{d_{\nu}}^{(1)}(t - mN_sT_f - C^{(1)}_m(t) - \tau^{(1)}_m) + n_{TOT}(t), \]

\( t \in T_m \triangleq [mN_sT_f + \tau^{(1)}, (m + 1)N_s - 1)T_f + \tau^{(1)}] \), where

\[ n_{TOT}(t) = \sum_{\nu=2}^{N_u} A^{(\nu)} x^{(\nu)}(t) + n(t) \]

includes both MA interference and thermal noise, and is assumed to be a mean-zero Gaussian random process. Standard techniques [7] can then be used to calculate the \( M \)-ary bit error probability \( P_e \) for coherent detection of the EC TH PPM signals. This \( P_e \) values is

\[ P_e(N_u) = \frac{M}{M-1} \left[ 1 - \int_{-\infty}^{\infty} (1 - Q[\xi + \sqrt{2 \log_2(M) \text{SNRb}_{\text{tot}}(N_u)}])^{M-1} \right] \]

(1)

where \( Q[\xi] \) is the Gaussian-tail integral, and [5]

\[ \text{SNRb}_{\text{tot}}(N_u) = \left[ \text{SNRb}_{\text{tot}}(1)^{-1} + \frac{1}{R_b \sum_{\nu=2}^{N_u} \left( \frac{A^{(\nu)}_s}{A^{(1)}_s} \right)^2} \right]^{-1} \]

(2)

is the output bit SNR that the desired user observe in the presence of \( N_u - 1 \) other users,

\[ \mu = \frac{[Fw(1 - \gamma_w(\tau_2))]^2}{2 \sigma_\Delta^2} \]

is a normalized SNR parameter which is defined in terms of the pulse shape \( w(t) \) and the time shift \( \tau_2 \),

\[ \sigma_\Delta^2 = \frac{1}{T_f} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} w(t-s) w(t - \tau_2) dt \right)^2 ds, \]

and

\[ \text{SNRb}_{\text{tot}}(1) = \frac{1}{\log_2(M)} \left( \frac{A^{(1)}_s}{N_s} \right)^{1-\lambda} \]

is equivalent to the output bit SNR that the desired user might observe in single-link communications. The substitution of the bit SNR \( \text{SNRb}_{\text{tot}}(N_u) \) in (2) into the bit error probability \( P_e(N_u) \) in (1) will provide the desired relation between bit error probability \( P_e \), number of users \( N_u \) and bit transmission rate \( R_b \).

B. Multiple-access degradation factor

Let's define \( \text{SNRb}_{\text{spec}} \) to be the specified operating bit SNR to achieve the desired probability of error. Recall that \( \text{SNRb}_{\text{spec}}(1) \) is the bit SNR value when only user one is active, and that \( \text{SNRb}_{\text{spec}}(N_u) < \text{SNRb}_{\text{spec}}(1) \) is the actual bit SNR when \( N_u \) users are active in the system.
The SNR_{b_{	ext{rec}}}(N_u) \geq \text{SNR}_{b_{	ext{spec}}}$ is the required value of SNR_{b_{	ext{rec}}}(1) that makes SNR_{b_{	ext{rec}}}(N_u) = \text{SNR}_{b_{	ext{spec}}}, so user one can still meet the specified value of bit error probability even when $N_u$ users are active. The value of SNR_{b_{	ext{rec}}}(N_u)$ can be calculated solving

$$\text{SNR}_{b_{	ext{rec}}}(N_u) = \frac{\text{SNR}_{b_{	ext{spec}}}}{1 + \text{SNR}_{b_{	ext{rec}}}(1)} \left[ 1 - \frac{\frac{\mu/Tr}{R_b \sum_{\nu=1}^{N_u} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2}{1 - \text{SNR}_{b_{	ext{spec}}} \left( \frac{\mu/Tr}{R_b \sum_{\nu=1}^{N_u} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2} \right)} \right]^{-1}.$$  

(3)

Notice that there is a limit on how large $\text{SNR}_{b_{	ext{spec}}}$ can be for a given number of users $N_u$. This maximum value can be found if we set $\text{SNR}_{b_{	ext{rec}}}(N_u)$ take on large values in (3) to get

$$\text{SNR}_{b_{	ext{lim}}}(N_u) = \lim_{\text{SNR}_{b_{	ext{rec}}}(N_u) \to \infty} \frac{\text{SNR}_{b_{	ext{spec}}}}{1 + \text{SNR}_{b_{	ext{rec}}}(1)} \left[ 1 - \frac{\frac{\mu/Tr}{R_b \sum_{\nu=1}^{N_u} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2}{1 - \text{SNR}_{b_{	ext{spec}}} \left( \frac{\mu/Tr}{R_b \sum_{\nu=1}^{N_u} \left( \frac{A^{(\nu)}_b}{A^{(1)}_b} \right)^2} \right)} \right]^{-1}.$$  

Notice that by decreasing $\text{SNR}_{b_{	ext{spec}}}$ we can increase $R_{\text{max}}$, or $N_{\text{max}}$. The limit on how small $\text{SNR}_{b_{	ext{spec}}}$ can be for a given number of users $N_u$ is investigated in the next subsection.

### C. Multiple-access transmission capacity

It is well known from communication theory that $P_e$ in (1) has the following limiting behavior $^5$

$$\lim_{M \to \infty} P_e = \begin{cases} 1, & \text{if } \text{SNR}_{b_{	ext{spec}}} < \log_2(2) \\ 0, & \text{if } \text{SNR}_{b_{	ext{spec}}} > \log_2(2) \end{cases}$$  

(9)

We can use the condition in (9) together with (7) to write

$$N_{\text{max}} < N_{\text{IR}} \triangleq \frac{1}{\log_2(2)} \frac{1}{R_0} \frac{\mu/Tr}{N_u - 1}.$$  

(10)

Hence, $N_{\text{IR}}$ is attainable, in principle, using block waveforms with $M \to \infty$. Similarly, we can use the condition in (9) together with (8) to write

$$R_{\text{max}} < C_{\text{IR}}(N_u) \triangleq \frac{1}{\log_2(2)} \frac{1}{R_0} \frac{\mu/Tr}{N_u - 1}.$$  

(11)

Hence, the term $C_{\text{IR}}(N_u)$ plays the role of multiple-access channel capacity per user of IR in bits per second. Another way to see this is by using Shannon’s formula for channel capacity $^7$

$$C(B) = B \log_2 (1 + \frac{1}{B} P_{\text{lim}}(N_u))$$

with bandwidth $B \triangleq \frac{1}{T_u}$ in the order of Gigahertz, and

$$P_{\text{lim}}(N_u) \triangleq R_0 \text{SNR}_{b_{\text{lim}}}(N_u)$$

playing the role of effective signal power to power noise density ratio. Hence

$$C(B) = B \log_2 (1 + \frac{1}{B} \frac{\mu/Tr}{N_u - 1})$$

$$= \frac{B}{\log(2)} \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{1}{B} \frac{\mu/Tr}{N_u - 1} \right)^k.$$  

$^5$In (9) we have assumed that $\text{SNR}_{b_{\text{rec}}}(1) = \text{SNR}_{b_{\text{spec}}}(N_u)$ so that the condition $\text{SNR}_{b_{\text{lim}}}(N_u) = \text{SNR}_{b_{\text{spec}}}$ can be met.
With $B$ in the order of Gigahertz, $\mu$ in the order of hundreds, $T_f$ in the order of hundreds of nanoseconds, and $N_u >> 1$, it is clear that

$$\left(\frac{1}{B} \frac{\mu / T_f}{N_u - 1}\right) < 0.01$$

and (11) can be approximated

$$C(B) \simeq \frac{1}{\log(2)} \frac{\mu / T_f}{N_u - 1} = C_{\text{in}}(N_u)$$

If $C_{IR}$ is the multiple-access capacity per user of IR in bits per second, then

$$C_{\text{TOT}} \triangleq N_u C_{IR} \simeq \frac{1}{\log(2)} \frac{\mu}{T_f} N_u >> 1,$$

plays the role of total multiple-access capacity of IR in bits per second and gives an upper bound on the total combined bit transmission rate that can be attained when the performance is determined by the amount of multiple-access interference with $N_u$ users active.

**IV. Numerical example**

In this section we evaluate $N_u(\text{DF})$ in (6) and $C_{\text{in}}(N_u)$ in (10) for a specific design. In IR modulation, the UWB received impulse $w(t)$ can be modeled by

$$w(t) = \left[1 - 4\pi \left(\frac{t}{t_n}\right)^2 \right] \exp \left(-2\pi \left(\frac{t}{t_n}\right)^2\right)$$

where the value $t_n$ (ns) is used to fit the model $w(t)$ to a measured waveform from a particular experimental IR link. The normalized signal correlation function corresponding to this impulse is

$$\gamma_w(t) = \left[1 - 4\pi \left(\frac{t}{t_n}\right)^2 + \frac{4\pi^2}{3} \left(\frac{t}{t_n}\right)^4 \right] \exp \left(-\pi \left(\frac{t}{t_n}\right)^2\right)$$

In this case $t_{\text{min}}$ (ns) depends on $t_n$, and $t_{\text{min}} = -0.6183$ for any $t_n$.

The value of $\mu$ was calculated for three different pulse widths $t_n$, using $\tau_2 = t_{\text{min}}$ and $T_f = 100$ ns. These values are shown in table I.

Figure 1 shows $N_u(\text{DF})$ for the EC TH PPM signal set, using $R_b = 1048$ Kilobits per second per user and $2 \leq M \leq 16$ with $P_e(1) \simeq 10^{-7}$. The curves were calculated using $\tau_2 = t_{\text{min}} = 0.2419$ ns and $T_f = 100$ ns. Perfect power control was assumed.

Figure 2 shows $N_u(\text{DF})$ for the EC TH PPM signal set, this time using $R_b = 9.6$ Kilobits per second per user and $2 \leq M \leq 1024$ with $P_e(1) \simeq 10^{-3}$. Also shown is the

**Fig. 1.** The number of users $N_u(\text{DF})$ for EC PPM signals, calculated using $2 \leq M \leq 16$ with $P_e(1) \simeq 10^{-7}$. The curves were calculated using $R_b = 1048$ Kilobits per second and set 2 of parameters in table I.

**Fig. 2.** The number of users $N_u(\text{DF})$ for EC PPM signals, calculated using $2 \leq M \leq 1024$ with $P_e(1) \simeq 10^{-3}$. Also shown is the value of $N_u(\text{DF}) \rightarrow N_{\text{IR}}$ for large values of both DF and $M$. The curves were calculated using $R_b = 9.6$ Kilobits per second and set 2 of parameters in table I.
value of $N_u(DF) \rightarrow N_{in}$ for large values of both DF and $M$. The curves were calculated using $\tau_2 = \tau_{min} = 0.2419$ ns and $T_f = 100$ ns. Again, perfect power control was assumed.

Figure 3 show the multiple access capacity of IR $C_{IR}(N_u)$ in bits per second corresponding to sets 1, 2, 3 of parameters in table 1.

![Graph showing multiple access capacity per user](image)

**Figure 3.** The multiple access capacity per user $C_{IR}(N_u)$ in bits per second as a function of $N_u$, calculated using sets 1, 2, 3 of parameters in table 1.

**V. Discussion of results**

Figures 1 and 2 shows how by increasing the value of $M$ it becomes possible to increase the maximum number of users $N_{max}$, for fixed values of the bit transmission rate and probability of bit error. In figure 2 the value $N_{max}$ is shown to reach a maximum $N_{in}$ no matter how large $M$ becomes.

From figure 3 it is clear that the multiple-access capacity per user $C_{IR}(N_u)$ using set 1 is larger than $C_{IR}(N_u)$ using set 2, which in turn is larger than $C_{IR}(N_u)$ using the set 3 of parameters in table 1. Similarly, the total multiple-access capacity $C_{TOT} = 3.3964$ Gigabits using set 1 is larger than $C_{TOT} = 2.3412$ Gigabits using set 2, which in turn is larger than $C_{TOT} = 1.3903$ Gigabits using the set 3 of parameters. This is to be expected since the set 1 corresponds to “more impulsive” signals, which means that the TH-PPM signals corresponding to different users are less likely to suffer collisions among them. From the frequency point of view, a narrower impulse implies more spreading gain in the frequency domain.

**References**


